vanishes for $u \neq 0$. Thus (7) has at most one positive solution, and similarly at most one negative solution.

Finally let $\alpha = 0$. Equations (4), (3) reduce to

$$3 \sin \beta l = \beta(1 + \cos \beta l),$$

$$\beta \sin \beta l = 3(1 - \cos \beta l);$$

and this pair of equations is equivalent to the single equation

$$3 \sin \tfrac{1}{2}\beta l = \beta \cos \tfrac{1}{2}\beta l.$$

Since it is clear that no value which makes $\cos \frac{1}{2}\beta l$ vanish can be a solution, the only admissible values of β are those for which

$$\tan \tfrac{1}{2}\beta l = \tfrac{1}{3}\beta, \tag{8}$$

which may be written

$$\tan u - ku = 0, \tag{9}$$

where

$$u = \tfrac{1}{2}\beta l, \qquad k = \frac{2}{3l}.$$

This is a type of equation which arises in many problems of analysis. Inspection of the graphs of $\tan u$ and ku leads at once to the well-known fact, which can readily be proved by rigorous analytic methods, that (8) has an infinite number of solutions, approximating more and more closely, as they increase numerically, to odd multiples of $\pi/2$.

We condense all these results into the statement: The function x^n, where n is a constant whose real part is positive, satisfies the proposed functional equation for the following values of n and no others:

(*a*) The real values $n = 1, 2, 3$.

(*b*) An infinite set of values of the form $n = 2 + \beta i$, where β is a solution[1] of (8).

Any of the second set of solutions leads to real solutions of the type considered by Professor Bennett. In fact, since

$$x^n = x^{m+2} = x^{\alpha+2+\beta i} = x^{\alpha+2} \cos(\beta \log x) + i x^{\alpha+2} \sin(\beta \log x),$$

for any n yielding a complex solution we have the two real solutions $x^{\alpha+2} \cos(\beta \log x)$ and $x^{\alpha+2} \sin(\beta \log x)$, whence also the more general solution $x^{\alpha+2} \sin(\beta \log x - c)$, which is a linear combination of them. Equations (3), (4) are equivalent to Professor Bennett's pair of equations for a, b with $a = \alpha + 2$, $b = \beta$. We have thus shown that Professor Bennett's formula gives a solution only (except for the trivial cases $b = 0$, $a = 1, 2, 3$) when $a = 2$, and that in this instance an infinite number of values of b will be possible. It is obvious that much more general solutions can be constructed, of the form $x^2 F(x)$, where

$$F(x) = \Sigma[A_p \cos(\beta_p \log x) + B_p \sin(\beta_p \log x)],$$

proper precautions being taken to insure the validity of the series.

The importance of these new solutions lies in the following facts. The function $x^2 \sin(\beta \log x - c)$ possesses a continuous first derivative even at the origin. In previous editorial comment, it was stated that "it seems reasonable to adopt as a goal with reference to this question the proof that if the equation holds for some range of the variable h, the function $F(x)$ can only be a polynomial of degree $\leqq 3$, under restrictions as light as possible—*e.g.*, that $F(x)$ should be continuous and possess a stated number (as small as it can be made) of derivatives." The statement has now been proved, with the number of derivatives as great as *six*, by Professor Gillespie. Professor Bennett's example, together with the present comments, shows that the theorem is not true with the number of derivatives as small as *one*. Can the gap between these two results be bridged?

DISCUSSIONS.

In a previous number of the MONTHLY (*1920*, 53) Mr. W. F. Cheney gave an instance of a geometric proof of the law of tangents in trigonometry. In the first discussion below Professor Lovitt shows how a number of such proofs may be obtained. The two papers contain references to a number of other similar proofs.

[1] The smallest positive value of β is approximately 12.94.

In the usual form of the law of the mean for derivatives, $f(x+h) - f(x) = hf'(x + \theta h)$, the quantity θ depends on both x and h. It is natural to inquire into the character of the functional dependence of θ on x and h. A general investigation of this dependence has been made by Hedrick, *Annals of Mathematics*, series 2, p. 177. Professor Downing in the second discussion undertakes the more special task of studying the nature of θ for simple forms of the function $f(x)$. He takes up the cases of linear, quadratic, and cubic functions.

I. Geometrical Proofs of the Law of Tangents.

By W. V. Lovitt, Colorado College.

The geometrical proof depends in many cases upon finding pairs of lines the ratio of whose lengths is $(a-b)/(a+b)$, where a and b are sides of a given triangle. We proceed then to find such pairs of lines.

Make

$$AC = SC = CV = CR = b;$$

make

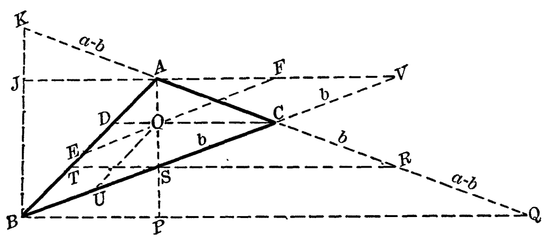
$$CK = CQ = CB = a.$$

Then

$$AK = BS = RQ = a - b,$$

and

$$BV = RK = a + b.$$



Draw CD bisecting the angle ACB. The lines VJ, TR, CD, and BQ are parallel. Draw AP and $BK \perp BQ$. Draw $EOF \parallel BC$ and $OU \parallel AB$.

Then

$$BU = \tfrac{1}{2}(a-b), \qquad UC = \tfrac{1}{2}(a+b),$$

and

$$\frac{a-b}{a+b} = \frac{BS}{BV} = \frac{EO}{EF} = \frac{BU}{UC} = \frac{DO}{OC} = \frac{TS}{AV} = \frac{TS}{SR} = \frac{BP}{PQ} = \frac{JA}{JV} = \frac{BS}{AQ} = \frac{AK}{BV}. \tag{1}$$

We also find

$$\tfrac{1}{2}(A+B) = \angle OSC = \angle OAC = \angle JBV = \angle JKA,$$

$$\tfrac{1}{2}(A-B) = \angle BAP = \angle JBA.$$

In many geometrical proofs of the law of tangents the only points of difference are in the precise figure and method used in establishing some one or more of the equalities (1). If equalities (1) were a well known part of the course in elementary geometry, then the several different proofs of the law of tangents would be simpler.

We give below some proofs which are believed to be new. With one excep-